2020 A

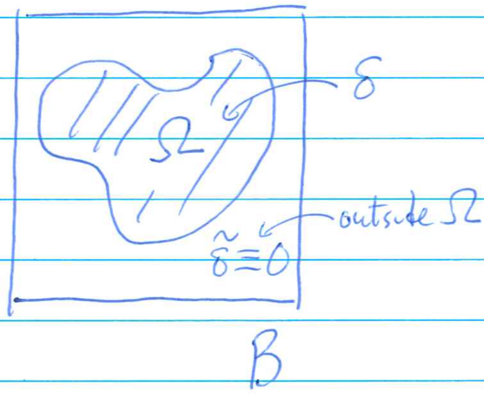
Lecture 8

Fall 2020

- Mass, Center of Mass, etc

Consider a solid occupying $\Omega \subseteq \mathbb{R}^3$ with density fcn δ. Its mass is approximately given by a Riemann sum

$$\sum \Delta m_{ijk} \quad \text{mass in each sub-rectangle box}$$



$$= \sum_{i,j,k} \tilde{\delta}(x_j^*, y_k^*, z_l^*)\, \Delta x_j \Delta y_k \Delta z_l$$

$$\to \iiint_B \tilde{\delta}(x,y,z)\, dV$$

$$= \iiint_\Omega \delta(x,y,z)\, dV.$$

So, we define the mass of the solid to be

$$\iiint_\Omega \delta(x,y,z)\, dV.$$

Next, center of mass. Consider 1-dim case first.

In discrete case, suppose $m_1, \dots, m_n$ are distributed at the position $x_1, \dots, x_n$. To find the balance point x_0 we need to solve

$$\sum_{x_j \geq x_0} m_j (x_j - x_0) = \sum_{x_j \leq x_0} m_j (x_0 - x_j), \text{ ie}$$

$$\sum_{j=1}^{n} m_j (x_j - x_0) = 0.$$

We readily get

$$x_0 = \frac{\sum_j m_j x_j}{\sum_j m_j}.$$

In the continuum case, it becomes a rod $[a, b]$ of density δ.

$$\sum_j \Delta m_j (x_j^* - x_0) = \sum_j \delta(x_j^*) \Delta x_j (x_j^* - x_0)$$

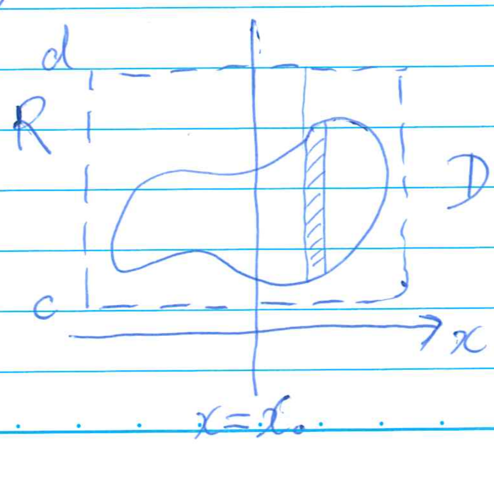
(x_j^* tag in a partition)

$$\rightarrow \int_a^b \delta(x)(x - x_0)\, dx = 0$$

$$x_0 = \frac{\int_a^b \delta(x)\, x\, dx}{\int_a^b \delta(x)\, dx}$$

or

$$= \frac{\int_a^b \delta(x)\, x\, dx}{M},$$ M mass of the rod.



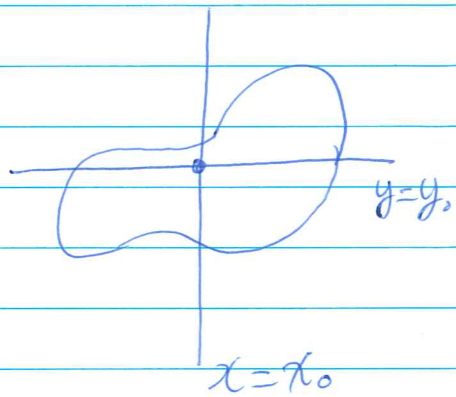
In 2-dim case,

$$\sum_j \int_c^d \tilde{\delta}(x_j^*, y)\, dy\, \Delta x_j (x_j^* - x_0)$$

$$\to \int_a^b \int_c^d \tilde{\delta}(x,y)(x-x_0)\,dx\,dy = 0, \text{ ie,}$$

$$\iint_D \delta(x,y)(x-x_0)\,dx\,dy = 0 \quad (\because \tilde{\delta} = 0 \text{ outside } D)$$

$$\therefore \quad x_0 = \frac{\iint_D \delta(x,y)\,x\,dx\,dy}{\iint_D \delta(x,y)\,dx\,dy}$$



Similarly,

$$y_0 = \frac{\iint_D \delta(x,y)\,y\,dx\,dy}{\iint_D \delta(x,y)\,dx\,dy}$$

The point (x_0, y_0) is the center of mass of D with density δ. Notations:

$$M_y = \iint_D \delta(x,y)\,x\,dx\,dy \qquad \text{first moment about } y\text{-axis}$$

$$M_x = \iint_D \delta(x,y)\,y\,dx\,dy \qquad \text{first moment about } x\text{-axis}$$

$$\left(\frac{M_y}{M}, \frac{M_x}{M}\right) \ (= (x_0, y_0)) \qquad \text{center of mass.}$$

when δ = constant, center of mass is called the centroid of D.

Similarly, in 3-dim

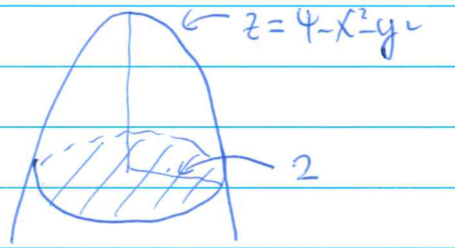
$$M_{yz} = \iiint_\Omega \delta(x,y,z)\, x\, dV$$ first moment about yz-plane

$$M_{xz} = \iiint_\Omega \delta(x,y,z)\, y\, dV$$ ---- about xz-plane

$$M_{xy} = \iiint_\Omega \delta(x,y,z)\, z\, dV$$ ---- about xy-plane.

center of mass : $\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right)$.

e.g. Let Ω be the solid bdd by $z=0$ and $z=4-x^2-y^2$. Find its center of mass. $(\delta \equiv 1)$



$D: x^2+y^2 \le 4$.

$$M = \iiint_\Omega dV = \iint_D \int_0^{4-x^2-y^2} dz\, dA(x,y)$$

$$= \iint_D (4-x^2-y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^2 (4-r^2)\, r\, dr\, d\theta$$

$$= 8\pi.$$

$$M_{xy} = \iiint_\Omega z\, dV = \iint_D \int_0^{4-x^2-y^2} z\, dz\, dA(x,y)$$

$$= \frac{1}{2} \iint_D (4 - x^2 - y^2)^2 \, dA$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)^2 \, r \, dr \, d\theta$$

$$\vdots$$

$$= \frac{32\pi}{3}.$$

$M_{yz} = M_{xz} = 0$ due to symmetry,

$\therefore$ center of mass

$$= \left(0, 0, \frac{32\pi/3}{8\pi}\right) = \left(0, 0, \frac{4}{3}\right).$$

Finally, the moment of inertia of solid Ω with density δ about an axis is defined to be

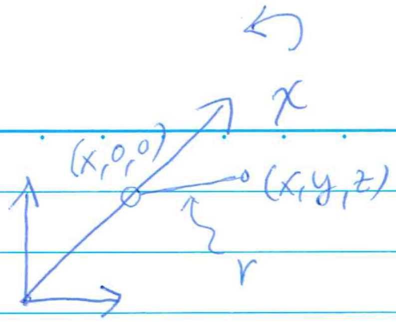
$$\iiint_\Omega \delta(x, y, z) \, r^2 \, dV,$$

where r is the distance from (x, y, z) to the axis. So,

$$I_x = \iiint_\Omega (y^2 + z^2) \, \delta(x, y, z) \, dV$$

$$I_y = \iiint_\Omega (x^2 + z^2) \, \delta(x, y, z) \, dV$$

$$I_z = \iiint_\Omega (x^2 + y^2) \, \delta(x, y, z) \, dV.$$



e.g. Find the moments of inertia for the ball of radius R. $(\delta \equiv 1)$

$$r = \|(x,y,z) - (x,0,0)\|$$
$$= \sqrt{y^2+z^2}.$$

$$I_z = \iiint_B (x^2+y^2)\,\delta(x,y,z)\,dV$$

$$= \iiint_B (x^2+y^2)\,dV$$

$$= 2\iint_D \int_0^{\sqrt{R^2-x^2-y^2}} (x^2+y^2)\,dz\,dA(x,y)$$

$$= 2\iint_D (x^2+y^2)\sqrt{R^2-x^2-y^2}\,dA$$

$$= 2\int_0^{2\pi}\int_0^R r^2\sqrt{R^2-r^2}\,r\,dr\,d\theta$$

$$\vdots$$

$$= \frac{8\pi}{15}R^2$$

$\left(M = \text{volume of } B = \frac{4}{3}\pi R^3\right)$

or

$$= \frac{2M}{5}R^2.$$

By symmetry, $I_x = I_y = \frac{2M}{5}R^2$. #